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Superfield theory and supermatrix model

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Abstract: We study the noncommutative superspace of arbitrary dimensions in a systematic way. Superfield theories on a noncommutative superspace can be formulated in two folds, through the star product formalism and in terms of the supermatrices. We elaborate the duality between them by constructing the isomorphism explicitly and relating the superspace integrations of the star product lagrangian or the superpotential to the traces of the supermatrices. We show there exists an interesting fine tuned commutative limit where the duality can be still maintained. Namely on the commutative superspace too, there exists a supermatrix model description for the superfield theory. We interpret the result in the context of the wave particle duality. The dual particles for the superfields in even and odd spacetime dimensions are D-instantons and D0-branes respectively to be consistent with the T-duality.

Keywords: Supersymmetry and Duality, Superspaces, Non-Commutative Geometry.

Contents

1. Introduction and summary

The wave particle duality was the key ingredient for the birth of the quantum theory. In the standard quantum field theory, the field represents the corresponding wave while the notion of the particle arises after the second quantization of the classical field. Remarkably in string/M-theory the wave particle duality is often manifest even at the classical level. The BFSS matrix model is essentially a description of the M-theory in terms of the infinitely many D-particles [1], while the IKKT matrix model formulates IIB superstring theory through the dynamics of the D-instantons [2]. Expanding the *matrix* models around the D-brane solution one can derive the supersymmetric Yang-Mills *field* theory action or the worldvolume description of the D-branes (see e.g. [3]–[5]). One characteristic feature of the matrix models is that they incorporate the identical nature of the particles since the permutation symmetry for the labelling of the particles is taken as the gauge symmetry [6, 7].

The close relation between the field theory and the matrix model was first studied in the early eighties by Eguchi and Kawai et al. [8, 9], and more recently by Dijkgraaf and Vafa showing that the effective superpotential in the four dimensional $\mathcal{N}=1$ supersymmetric gauge theory can be read off from the associated matrix model computation [10].

The purpose of the present paper is to show directly that there exits a dual supermatrix model for the low energy description of any superfield theory. Our scheme is first to introduce the noncommutativity on the superspace [11]–[20] replacing all the ordinary products of the superfields by the generalized Moyal-Weyl star products. The theory is then equivalent to the operator formalism on a noncommutative superspace or the supermatrix model description. A remarkable fact is that, contrary to the nonsupersymmetric case, the equivalence between the superfield theory and the supermatrix model is still maintained in a certain commutative limit. Thus, turning off the noncommutativity in a fine tuned manner we are able to recover the original commutative superfield theory accompanied by the dual supermatrix model.[1]

The main content of our paper is to establish precisely the equivalence between the star product formalism and the supermatrix description. The organization of the paper is as follows.

In section 2, we first consider the most general noncommutative relations on the superspace. In contrast to the work by Seiberg on the noncommutative Euclidean superspace [13], we have in mind the study of the real action in the minkowskian spacetime so that we consider all the possible noncommutativity among all the coordinates on the superspace. In particular, we first consider the real basis for the superspace coordinates. Namely, for the complex Grassmann coordinates in the four dimensional chiral superspace, we decompose them into the real ones. Analyzing the real basis we are able to obtain the general unitarity representations of the noncommutative superspace.

In section 3, we show in detail how the star product formalism on a commutative superspace realizes the noncommutative superspace. Explicitly through the generalized Weyl ordering prescription, we construct the isomorphism between the star product on the commutative superspace and the operator product on the noncommutative superspace.

In section 4, we study the supermatrix representation of the operators on the noncommutative superspace. To do so we first find the 'canonical' form for the noncommutativity parameters under the linear transformations of the super-coordinates. We show that the generic noncommutative superspace decomposes into two sub-superspaces which commute each other. One consists of the bosonic and fermionic harmonic oscillators having the nondegenerate noncommutativity parameters, while on the other sub-superspace, the only possible noncommutative relation is given by the commutator between the even and odd coordinates. Surely the bosonic harmonic oscillators are represented by matrices of the infinite size and the fermionic oscillators or the Clifford algebra have the gamma matrix representation. For the second sub-superspace we restrict on the commutative sub-superspace and continue the analysis for simplicity. The superfields on the noncommutative superspace are then represented by supermatrices. Depending on the statistics of the superfields the corresponding supermatrices have the different structures. In particular, the supermatrix representation of the ordinary Grassmann number is given by the "gamma five" matrix. We also study the noncommutative chiral superspace by complexifying the odd coordinates. We point out that the anti-commutator between the odd coordinate and its complex conjugate must be nonzero in the generic nontrivial cases.

[1]After submitting the first version of our paper, we learned that [15] contains the similar idea but differs in details.

In section 5, we rewrite the integration of the star product lagrangians over the superspaces as the traces of the supermatrix lagrangians. The integration over the full superspace is essentially given by the supertrace, while the integration over the chiral superspace involves taking trace of the superpotential matrix times a certain gamma matrix. We also present formulae which relate the partition function and the correlation functions in the star product superfield theory to those in the corresponding supermatrix model. Finally we argue that there exit an interesting commutative limit where the duality between the superfield theory and the supermatrix model does not break down. In particular, in the commutative limit, the kinetic part of the supermatrix model tends to diverge so that it is natural to use the saddle point analysis imposing the constraint on the supermatrices which sets the divergent part to vanish. Hence, the saddle point approximation essentially gives the low energy description.

In section 6, we explicitly study two examples, $\mathcal{N}=1, D=3$ nonlinear sigma model and $\mathcal{N}=1, D=4$ chiral model. In general, turning on the noncommutativity on the superspace breaks the supersymmetry. However, we show that for a certain 'minimally' noncommutative superspace, the supersymmetry is not broken in the chiral model.

Section 7 contains the comment on the nonexistence of the $\mathcal{N}=4, D=4$ superfield formalism, and the appendix carries our proof of the isomorphism between the star product on the commutative superspace and the operator product on the noncommutative superspace.

2. Noncommutative superspace

Before we start, it is useful to define the following commutator or anti-commutator for the superspace coordinates, z^A,

$$[z^A, z^B\}_\pm \equiv z^A z^B \pm (-1)^{\#_A \#_B} z^B z^A \,, \tag{2.1}$$

where $\#_A$ is the Z_2-grading of the superspace so that $\#_A = 0$ for even or bosonic A, and $\#_A = 1$ for odd or fermionic A. The ordinary commutative superspace satisfies simply,

$$[z^A, z^B\}_- = 0 \,. \tag{2.2}$$

Any deformation of the above relation defines a noncommutative superspace on which the coordinates, $\hat{z}^A$, are operators and satisfy the noncommutative relation,

$$[\hat{z}^A, \hat{z}^B\}_- = \Omega^{AB} \,. \tag{2.3}$$

The noncommutative parameter, Ω^{AB}, is taken here to be constant. Since the left and right hand sides must agree for the Z_2-grading, $\#_A + \#_B \sim \#_{\Omega^{AB}}$: mod 2. Note also

$$\Omega^{AB} + (-1)^{\#_A \#_B} \Omega^{BA} = 0 \,. \tag{2.4}$$

Explicitly with the even and odd coordinates, $\hat{z}^\mu$, $\hat{z}^\alpha$,

$$[\hat{z}^\mu, \hat{z}^\nu] = i\Theta^{\mu\nu} \,, \qquad [\hat{z}^\mu, \hat{z}^\alpha] = i\Psi^{\mu\alpha} \,, \qquad \{\hat{z}^\alpha, \hat{z}^\beta\} = S^{\alpha\beta} \,, \tag{2.5}$$

so that

$$\Omega = \begin{pmatrix} i\Theta & i\Psi \\ -i\Psi^T & S \end{pmatrix} . \tag{2.6}$$

The superdeterminant of Ω is then[2]

$$\mathrm{sdet}(\Omega) = \frac{\det(i\Theta - \Psi S^{-1}\Psi^T)}{\det(S)} = \frac{\det(i\Theta)}{\det(S + i\Psi^T\Theta^{-1}\Psi)} . \tag{2.7}$$

Henceforth, otherwise mentioned explicitly, we consider the real basis for the superspace with the reality condition, $(\hat{z}^A)^\dagger = \hat{z}^A$. Taking the real basis will enable us to analyze the most general unitary representations of the noncommutative superspace as done in section 4. In particular, the four dimensional chiral superspace will be treated as the complexification of the corresponding real superspace. In the real basis the noncommutative parameters, $\Theta^{\mu\nu}$, $\Psi^{\mu\alpha}$, $S^{\alpha\beta}$ are all real.

3. Star product formalism

In this section, we show in detail how the star product formalism on a commutative superspace realizes the noncommutative superspace. We first define a one to one map, $\mathcal{O}$, from the commutative superspace to the noncommutative superspace by the generalized Weyl ordering,

$$\mathcal{O}(z^{A_1}z^{A_2}\cdots z^{A_n}) \equiv \hat{z}^{[A_1}\cdots\hat{z}^{A_n\}+} = \sum_{P=1}^{n!} \frac{(-1)^{N_P}}{n!}\,\hat{z}^{P_1}\cdots\hat{z}^{P_n} , \tag{3.1}$$

where P denotes the permutation of $(A_1, \cdots, A_n)$ and N_P counts the number of permuted times only for the odd coordinates in $(z^{A_1}\cdots z^{A_n})$. The star ($\star$) product of the superfields on the commutative superspace reads with the noncommutative parameter, Ω^{AB},

$$f(z) \star g(z) \equiv f(z)\, e^{\frac{1}{2}\overleftrightarrow{\Omega}}\, g(z) , \tag{3.2}$$

where explicitly in the expansion of the exponential,

$$\left(\overleftrightarrow{\Omega}\right)^n = \overleftarrow{\partial}_{A_1}\overleftarrow{\partial}_{A_2}\cdots\overleftarrow{\partial}_{A_n}(\Omega^{A_nB_n}\overrightarrow{\partial}_{B_n})\cdots(\Omega^{A_2B_2}\overrightarrow{\partial}_{B_2})(\Omega^{A_1B_1}\overrightarrow{\partial}_{B_1}) , \tag{3.3}$$

which is consistent with the fact that $\overleftrightarrow{\Omega}$ is even. The right and left derivatives are related by (cf. [13])

$$f\overleftarrow{\partial}_A = (-1)^{\#_A(\#_f+1)}\overrightarrow{\partial}_A f , \tag{3.4}$$

so that $z^B\overleftarrow{\partial}_A = \overrightarrow{\partial}_A z^B = \delta_A{}^B$.

The duality between the operator and the star product formalisms comes from the following crucial isomorphism,

$$\mathcal{O}(f)\mathcal{O}(g) = \mathcal{O}(f \star g) , \tag{3.5}$$

which also manifests the associativity of the star product. The appendix carries our proof on the isomorphism.

[2]The last equality in eq. (2.7) comes from the identity, $\det(1+i\Theta^{-1}\Psi S^{-1}\Psi^T) = \det^{-1}(1+iS^{-1}\Psi^T\Theta^{-1}\Psi)$, which can be easily shown using $\ln(\det M) = \mathrm{tr}(\ln M)$.

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The derivatives act on the noncommutative superspace as

$$\mathcal{O}(\partial_A f) = [\hat{\partial}_A, \mathcal{O}(f)\}_- \,, \qquad \hat{\partial}_A \equiv (\Omega^{-1})_{AB}\hat{z}^B \,. \tag{3.6}$$

It is also worth to note

$$z^A f(z) = \frac{1}{2}[z^A, f(z)\}_{\star +} = \frac{1}{2}\left(z^A \star f(z) + (-1)^{\#_A \#_f} f(z) \star z^A\right) . \tag{3.7}$$

4. Supermatrix representation

Under the linear transformation, $\hat{z}^A \rightarrow \hat{z}'^A = P^A{}_B \hat{z}^B$, the noncommutative parameter, Ω, transforms[3] to $\Omega' = P\Omega P^T$. Acting a proper transformation one can obtain its 'canonical' form. Firstly, for $\det(\Theta)\det(S) \neq 0$, we consider the following transformation which preserves the reality, $(\hat{z}'^A)^\dagger = \hat{z}'^A$,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 + \displaystyle\sum_{n=0}^{\infty} a_n (i\Psi^T\Theta^{-1}\Psi S^{-1})^n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Psi^T\Theta^{-1} & 1 \end{pmatrix} , \tag{4.1}$$

where the coefficients are given by $a_0 = 0$, $a_1 = -1/2$ and

$$a_{n+1} = -a_n - \frac{1}{2}\sum_{j=1}^{n} a_j(a_{n+1-j} + a_{n-j}) \,, \qquad \text{for } n \geq 1 \,. \tag{4.2}$$

From the Grassmann property of Ψ, the sum in eq. (4.1) is actually a finite one. The resulting noncommutative parameter is then block diagonal, $\Omega' = \begin{pmatrix} i\Theta & 0 \\ 0 & S \end{pmatrix}$, as the odd parameter, Ψ, is now completely removed. In the case of $\det(\Theta)\det(S) = 0$ too, one can apply the previous analysis to the nondegenerate sector and achieve the canonical form. The final result for the generic cases can be summarized as follows. Any noncommutative superspace, $\{\hat{z}^A\}$, decomposes into two sub-superspaces which commute each other,

$$\begin{aligned} \{\hat{z}^A\} &= \{\hat{z}^M = (\hat{x}^\mu, \hat{\vartheta}^a)\} \oplus \{\hat{w}^{\dot{K}} = (\hat{y}^{\dot{\mu}}, \hat{\eta}^{\dot{a}})\} \,, \\ (\hat{z}^M)^\dagger &= \hat{z}^M \,, \qquad (\hat{w}^{\dot{K}})^\dagger = \hat{w}^{\dot{K}} \,, \qquad [\hat{z}^M, \hat{w}^{\dot{K}}\} = 0 \,. \end{aligned} \tag{4.3}$$

Each sub-superspace satisfies

$$\begin{aligned} &[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu} \,, \qquad \{\hat{\vartheta}^a, \hat{\vartheta}^b\} = s^{ab} \,, \qquad [\hat{x}^\mu, \hat{\vartheta}^a] = 0 \,, \\ &[\hat{y}^{\dot{\mu}}, \hat{y}^{\dot{\nu}}] = 0 \,, \qquad \{\hat{\eta}^{\dot{a}}, \hat{\eta}^{\dot{b}}\} = 0 \,, \qquad [\hat{y}^{\dot{\mu}}, \hat{\eta}^{\dot{a}}] = i\chi^{\dot{\mu}\dot{a}} \,, \end{aligned} \tag{4.4}$$

where all the parameters are real, and $\theta^{\mu\nu}$, s^{ab} are nondegenerate, $\det(\theta)\det(s) \neq 0$. Furthermore, as $\theta^{\mu\nu}$ and s^{ab} are respectively anti-symmetric real and positive definite symmetric real matrices, there exit real linear transformations which take them to the canonical forms,

$$\theta^{\mu\nu} = \mathrm{diag}\,(\epsilon, \epsilon, \cdots, \epsilon) \,, \qquad \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad s^{ab} = \delta^{ab} \,. \tag{4.5}$$

[3]The transpose of the supermatrix is defined as $(P^T)_A{}^B = (-1)^{\#_A(\#_A + \#_B)} P^B{}_A$ [21].

Namely, $\hat{x}^\mu$'s satisfy the Heisenberg algebra, while $\hat{\vartheta}^a$'s satisfy the Euclidean Clifford algebra. Their unitary irreducible representations are unique and very well known, the harmonic oscillator type infinite matrices and the gamma matrices respectively. As for the second sub-superspace, $\{\hat{w}^{\dot{K}}\}$, if its only noncommutative parameter vanishes, $\chi^{\dot{\mu}\dot{a}}=0$, the sub-superspace becomes simply a commutative one. However, for the generic case, $\chi^{\dot{\mu}\dot{a}}\neq 0$, we do not know the unitary representation, and henceforth we restrict on the commutative case, $\chi^{\dot{\mu}\dot{a}}=0$. Furthermore, without loss of generality for any spacetime dimensions, we take the dimension of the noncommutative odd coordinates to be even, $\hat{\vartheta}^a$, $1\leq a\leq N$, $N=$ even. For example, for the $\mathcal{N}$-extended supersymmetry in the minkowskian four dimensions, $N=4\mathcal{N}$. As the gamma matrices are then taken from the Euclidean even dimensions, they can be set hermiean and off-block diagonalized. The irreducible supermatrix representation of the noncommutative superspace coordinates as well as the bosonic and fermionic component fields, $\phi(\hat{x})$, $\psi(\hat{x})$, are given by the direct product of the harmonic oscillator matrix and the gamma matrix,

$$\begin{array}{ll}\hat{\vartheta}^a\Longrightarrow 1\otimes\begin{pmatrix}0&\rho^a\\(\rho^a)^\dagger&0\end{pmatrix}, & \rho^a(\rho^b)^\dagger+\rho^b(\rho^a)^\dagger=s^{ab}\,,\\ \hat{x}^\mu\Longrightarrow\hat{x}^\mu\otimes 1\,, & w^{\dot{K}}\Longrightarrow w^{\dot{K}}\otimes\begin{pmatrix}1&0\\0&(-1)^{\#_{\dot{K}}}\end{pmatrix},\\ \mathcal{O}(\phi(x))\Longrightarrow\phi(\hat{x}^\mu)\otimes 1\,, & \mathcal{O}(\psi(x))\Longrightarrow\psi(\hat{x}^\mu)\otimes\begin{pmatrix}1&0\\0&-1\end{pmatrix}.\end{array}\tag{4.6}$$

Thus, in general, the supermatrix representation of the superfield, $\hat{F}(\hat{z})$, on the noncommutative superspace is of the form,

$$\hat{F}(\hat{z})=\sum\hat{F}_{[\vec{m},g,r]\,[\vec{n},h,s]}\,|\vec{m},g,r\rangle\langle\vec{n},h,s|\,,\tag{4.7}$$

where $[\vec{m},g,r]$ forms *an* index of the supermatrix such that $\vec{m}=(m_1,m_2,\cdot\cdot)$ are the non-negative integers for the harmonic oscillators, g denotes the gamma matrix index, and r is from the preexisting gauge group, if any [22]. Surely the symmetric products of the infinite matrices, $\hat{x}^\mu$, the anti-symmetric products of the gamma matrices and the gauge group generators give the basis for the supermatrix representation. As the superfield is an eigenstate of the Z_2-grading, the block diagonal and off-block diagonal components in the corresponding supermatrix have the opposite Z_2-gradings, and hence the name 'supermatrix',

$$\begin{pmatrix}\text{even}&\text{odd}\\\text{odd}&\text{even}\end{pmatrix}:\ \text{bosonic supermatrix}\,,\qquad\begin{pmatrix}\text{odd}&\text{even}\\\text{even}&\text{odd}\end{pmatrix}:\ \text{fermionic supermatrix}\,.\tag{4.8}$$

The coefficients of the supermatrix may depend on the commutative coordinates, $w^{\dot{K}}$, if any. It is also worth to note that for an arbitrary Grassmann number, ε,

$$\left[\begin{pmatrix}\varepsilon&0\\0&-\varepsilon\end{pmatrix},\ \text{supermatrix}\,\right\}_{-}=0\,,\tag{4.9}$$

which is consistent with $[\varepsilon,\,\text{superfield}\,\}_{-}=0$.

Complexifying the fermionic coordinates, the noncommutative chiral superspace reads, $(\hat{x}^\mu, \hat{\theta}^\alpha, \hat{\bar{\theta}}^{\dot{\alpha}} = (\hat{\theta}^\alpha)^\dagger)$, $1 \leq \alpha \leq N/2$, satisfying

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}\,, \qquad \{\hat{\theta}^\alpha, \hat{\theta}^\beta\} = c^{\alpha\beta}\,, \qquad \{\hat{\theta}^\alpha, \hat{\bar{\theta}}^{\dot{\beta}}\} = \tilde{c}^{\alpha\dot{\beta}}\,, \qquad \{\hat{\bar{\theta}}^{\dot{\alpha}}, \hat{\bar{\theta}}^{\dot{\beta}}\} = \bar{c}^{\dot{\alpha}\dot{\beta}}\,. \tag{4.10}$$

As shown above, in the supermatrix formulation of the noncommutative superfield theories, the fermionic coordinates are represented by the Euclidean N-dimensional gamma matrices so that for the complex odd coordinates,

$$\hat{\theta}^\alpha = P^\alpha{}_a \Gamma^a\,, \qquad \{\Gamma^a, \Gamma^b\} = \delta^{ab}\,, \qquad (\Gamma^a)^\dagger = \Gamma^a\,, \tag{4.11}$$

where $1 \leq \alpha \leq N/2$, $1 \leq a \leq N$ and $P^\alpha{}_a$ is a complex number. It follows that $\tilde{c}^{\alpha\dot{\beta}} = P^\alpha{}_a (P^\beta{}_a)^*$, etc. In particular, this implies that one can not simply switch off $\tilde{c}^{\alpha\dot{\beta}}$ as it would imply $P^\alpha{}_a = \hat{\theta}^\alpha = 0$. We find, up to the complex linear transformations, the canonical choice gives the conventional fermionic harmonic oscillators,

$$P^\alpha{}_a = \frac{1}{\sqrt{2}}(\delta^{2\alpha-1}{}_a + i\delta^{2\alpha}{}_a)\,, \qquad \text{so that} \qquad \tilde{c}^{\alpha\dot{\beta}} = \delta^{\alpha\dot{\beta}}\,, \qquad c^{\alpha\beta} = 0\,, \qquad \bar{c}^{\dot{\alpha}\dot{\beta}} = 0\,, \tag{4.12}$$

which we will call the 'minimally' noncommutative chiral superspace henceforth. As for the gamma matrix representation of the $N = 2$ case, we take $\hat{\theta} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $\hat{\bar{\theta}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, and for $N = 4, 6, \ldots$, we can construct the representation through the recurrent relation,

$$\hat{\theta}^\alpha \Leftarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \hat{\theta}^\alpha\,, \qquad \hat{\theta}^{\frac{N}{2}+1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes 1\,. \tag{4.13}$$

Unless mentioned otherwise, the minimal form is not to be assumed below.

In the minkowskian four dimensions, with the standard convention, $x^\mu_\pm = x^\mu \pm i\theta\sigma^\mu\bar{\theta}$, the chiral superspace normally means $\{x_+, \theta\}$ rather than $\{x, \theta\}$. However, in that case the star product of chiral superfields does not preserve the chirality. Thus, what we mean by the chiral superspace is strictly $\{x, \theta\}$ in the present paper which was also the case in [15]. We will come back to the chiral superspace in section 6.

5. Supermatrix action and the commutative limit

Now, with the map from the commutative superspace to the noncommutative superspace, (3.1), (4.6), the superfield action reads

$$\begin{aligned} \int dx^D d\vartheta^N \mathcal{L}_\star &= \frac{\sqrt{(2\pi)^D 2^{-N} \det(\theta)}}{\det(s)} \operatorname{Tr}\left[\hat{\vartheta}^{[N} \cdots \hat{\vartheta}^2 \hat{\vartheta}^{1]} \mathcal{O}(\mathcal{L}_\star)\right] \\ &= \sqrt{(-2\pi i)^D 2^{-N} \operatorname{sdet}(\Omega)}\, \operatorname{Tr}\left[\Gamma_{(N+1)} \mathcal{O}(\mathcal{L}_\star)\right], \end{aligned} \tag{5.1}$$

where $\mathcal{L}_\star$ denotes the superfield lagrangian equipped with the star product. $\Gamma_{(N+1)}$ is a normalized matrix or the generalized "gamma five",

$$\Gamma_{(N+1)} = \det(s)^{-1/2} \hat{\vartheta}^{[N} \cdots \hat{\vartheta}^2 \hat{\vartheta}^{1]}\,, \qquad (\Gamma_{(N+1)})^2 = (-1)^{\frac{1}{2}N(N-1)}\,, \tag{5.2}$$

and hence, $\operatorname{Tr}[\Gamma_{(N+1)}\, \cdot\,]$ is essentially $\sqrt{(-1)^{\frac{1}{2}N(N-1)}}$ times the supertrace.

The partition function in the superfield theory is related to that of the supermatrix model,

$$\int DF\, e^{i\int dz^{D+N}\mathcal{L}_\star(F)} = \mathrm{sdet}\Big(\frac{\partial F}{\partial \hat{F}}\Big) \int D\hat{F}\, e^{i\sqrt{(-2\pi i)^D 2^{-N}\mathrm{sdet}(\Omega)}\,\mathrm{Tr}[\Gamma_{(N+1)}\mathcal{L}(\hat{F})]}\,, \tag{5.3}$$

where $\hat{F} = \mathcal{O}(F(z))$ is the supermatrix. Since $\mathcal{O}$ is a linear map the jacobian is constant, and we are able to take it out from the integration. Furthermore, from the expression of the partition function above, the correlators of the superfields can be obtained from the supermatrix correlators [23],

$$\langle F(z_1)F(z_2)\rangle = \langle \hat{F}_{LM}\hat{F}_{JK}\rangle \Lambda^{LM}(z_1)\Lambda^{JK}(z_2)\,, \tag{5.4}$$

where L, M denote the indices of the supermatrix, $\hat{F}$, while $\Lambda^{LM}(z)$ forms a basis for the functions on the superspace satisfying $\mathcal{O}(\Lambda^{LM})_{JK} = \delta^L_J\,\delta^M_K$ so that $F(z) = \hat{F}_{LM}\Lambda^{LM}(z)$ and $\mathcal{O}(\hat{F}_{LM}\Lambda^{LM}) = \hat{F}$.

The conventional supersymmetry operator, Q_a, contains not only ∂_a but also a term like $\vartheta^a\partial_\mu$. From $\vartheta^a\partial_\mu = (\vartheta^a\star)\,\partial_\mu - \frac{1}{2}\Omega^{aA}\partial_A\partial_\mu$, it is clear that the supersymmetry operator does not satisfy the Leibniz rule anymore, and hence the supersymmetry is generically broken on the noncommutative superspace. Nevertheless, as we will show in the following section, for the 'minimally' noncommutative chiral superspace, the full supersymmetry can remain unbroken.

The integration over the chiral superspace reads

$$\int dx^D d\theta^{N/2} W_\star(x,\theta) = \sqrt{\frac{(2\pi)^D\det\theta}{2^N\det(\tilde{c})}}\,\mathrm{Tr}\,[\Gamma_\theta\mathcal{O}(W_\star)], \tag{5.5}$$

where Γ_θ is a normalized matrix,[4]

$$\Gamma_\theta = \det(\tilde{c})^{-1/2}\hat{\bar{\theta}}^{[N/2}\cdots\hat{\bar{\theta}}^2\hat{\bar{\theta}}^{1]}\,, \qquad \mathrm{tr}\left(\Gamma_\theta(\Gamma_\theta)^\dagger\right) = 2^{N/2}\,. \tag{5.6}$$

A remarkable fact about eq. (5.1) is that there exists a fine tuned commutative limit, since $\mathrm{sdet}(\Omega) = \det(i\theta)/\det(s)$ can remain fixed. This also implies that the jacobian factor in the partition function expression (5.3) converges to a nonzero constant, which would not be true for the bosonic noncommutative theory. In the commutative limit, the potential remains fixed but the kinetic term diverges as it contains the derivatives. The prescription for the supermatrix partition function is then to take the saddle point analysis with the constraint on the supermatrices such that the divergent part of the kinetic term vanishes. Similarly for the chiral theories, one can take the commutative limit where $\det(i\theta)/\det(\tilde{c})$ remains fixed and impose a constraint on the supermatrices which prevents the kinetic term from diverging. In this way, there exits a supermatrix description for any superfield theory

[4]Alternatively one can take the anti-symmetric product of $\hat{\partial}_\alpha$'s as Γ_θ with a suitable normalization constant, which would pick up the coefficient of the $\theta^{N/2}$ term only even for the integration over the generic nonchiral superfield, $W(x,\theta,\bar{\theta})$. However, restricting on the chiral superpotential, $W(x,\theta)$, of which the component field for the highest order in θ is bosonic, the above choice of Γ_θ serves the desired property well.

on a commutative superspace too. The supermatrix model lagrangian is same as that of the superfield theory, but the supermatrices are surely of infinite size meaning the planar limit. The dual particles for the superfields in even and odd dimensions are D-instantons and D0-branes respectively to be consistent with the T-duality.

6. Examples

6.1 $\mathcal{N}=1$ $D=3$ nonlinear sigma model

In the minkowskian three dimensions, the $\mathcal{N}=1$ superspace consists of $\{x^\mu,\vartheta^\alpha\}$ where $\alpha=1,2$ and the odd coordinates are Majorana spinors, $(\vartheta^\alpha)^\dagger=\vartheta^\alpha$. The generic nonlinear sigma model reads with the real superfields, $\Phi^i=\phi^i(x)+i\vartheta^\alpha\psi^i_\alpha(x)+i\vartheta^1\vartheta^2F^i(x)$, [24]

$$\int dx^3d\vartheta^2\ -\frac{1}{2}g_{ij}(\Phi)D^\alpha\Phi^iD_\alpha\Phi^j\,, \tag{6.1}$$

where $D_\alpha=\partial_\alpha+(\epsilon\gamma^\mu\vartheta)_\alpha\partial_\mu$, $D^\alpha=\epsilon^{-1\alpha\beta}D_\beta$ and ϵ is the 2×2 anti-symmetric matrix satisfying $\epsilon\gamma^\mu\epsilon^{-1}=-(\gamma^\mu)^T$. The supersymmetry transformation is given by $\delta\Phi^i=\varepsilon^\alpha Q_\alpha\Phi^i$ where ε^α is a real spinor and $Q_\alpha=\partial_\alpha-(\epsilon\gamma^\mu\vartheta)_\alpha\partial_\mu$.

From (3.7) we note $D_\alpha\Phi^i=\partial_\alpha\Phi^i+\frac{1}{2}\{(\epsilon\gamma^\mu\vartheta)_\alpha,\partial_\mu\Phi^i\}_\star$. With this expression we replace all the ordinary products by the star products to turn on the noncommutativity on the superspace,

$$[\hat{x}^1,\hat{x}^2]=i\lambda^2\,,\qquad\{\hat{\vartheta}^\alpha,\hat{\vartheta}^\beta\}=\lambda^2\delta^{\alpha\beta}\,, \tag{6.2}$$

where λ is a real parameter we introduce to control the strength of the noncommutativity so that $\hat{x}^1\sim\hat{x}^2\sim\hat{\vartheta}^\alpha\sim\lambda$. Note that the time coordinate, t, commutes with all other coordinates. In particular, using the pauli matrices we set $\hat{\vartheta}^1=\lambda\sigma^1$, $\hat{\vartheta}^2=\lambda\sigma^2$. The action on the noncommutative superspace reads

$$\int dx^3d\vartheta^2\ -\frac{1}{2}g_{\star ij}(\Phi)\star(D^\alpha\Phi^i)\star(D_\alpha\Phi^j)=i\pi\frac{1}{2}\int dt\,\mathrm{Str}\Big[g_{ij}(\hat{\Phi})(\hat{D}^\alpha\hat{\Phi}^i)(\hat{D}_\alpha\hat{\Phi}^j)\Big], \tag{6.3}$$

where Str denotes the supertrace, $\mathrm{Tr}[\sigma^3\cdot]$, and from (4.6) Φ^j's are generically time dependent hermitean supermatrices. Their covariant derivatives are of the form,

$$\hat{D}_\alpha\hat{\Phi}=\lambda^{-1}[\sigma^\alpha,\hat{\Phi}]+\frac{1}{2}\lambda(\epsilon\gamma^0)_{\alpha\beta}\{\sigma^\beta,\partial_0\hat{\Phi}\}+\frac{1}{2}(\epsilon\gamma^l)_{\alpha\beta}\{\sigma^\beta,[\hat{n}_l,\hat{\Phi}]\}\,, \tag{6.4}$$

where $\hat{n}_1=i\hat{x}^2/\lambda$, $\hat{n}_2=-i\hat{x}^1/\lambda$ are the normalized harmonic oscillators, $[\hat{n}_1,\hat{n}_2]=-i$.

The supersymmetry is surely completely broken. From (3.7), (4.6) and (4.9) with the rescaling, $\varepsilon\rightarrow\lambda\varepsilon\sigma^3$, the broken supersymmetry is given by

$$\delta_{rmbroken}\hat{\Phi}=[\varepsilon^\alpha\sigma^3\sigma^\alpha,\hat{\Phi}]-\frac{1}{2}\lambda^2(\epsilon\gamma^0)_{\alpha\beta}\{\varepsilon^\alpha\sigma^3\sigma^\beta,\partial_0\hat{\Phi}\}-\frac{1}{2}\lambda(\epsilon\gamma^i)_{\alpha\beta}\{\varepsilon^\alpha\sigma^3\sigma^\beta,[\hat{n}_i,\hat{\Phi}]\}\,. \tag{6.5}$$

In the commutative limit, $\lambda\rightarrow0$, from (6.4) the kinetic term tends to diverge. The saddle point prescription requires then $[\sigma^\alpha,\hat{\Phi}^i]=0$, and hence $\hat{\Phi}^i=\hat{\phi}^i(\hat{x})\otimes1_{2\times2}$. At the saddle point the action reduces to the bosonic matrix action,

$$-2\pi\int dt\,\mathrm{tr}\Big(g_{ij}(\hat{\phi})[\hat{n}_l,\hat{\phi}^i][\hat{n}_l,\hat{\phi}^j]\Big). \tag{6.6}$$

From (6.5), the supersymmetry is trivially "restored" as $\delta\Phi=i[\varepsilon^\alpha\sigma_\alpha,\hat{\Phi}]=0$.

6.2 $\mathcal{N}=1$ $D=4$ chiral theories — unbroken SUSY

In the minkowskian four dimensions, the $\mathcal{N}$-extended superspace reads $\{x^\mu, \theta^{a\alpha}, \bar\theta_a^{\dot\alpha}\}$, where $\bar\theta_a^{\dot\alpha} = (\theta^{a\alpha})^\dagger$, $\alpha, \dot\alpha = 1,2$ and $1 \leq a \leq \mathcal{N}$ so that the dimension of the fermionic real coordinates is $N = 4\mathcal{N}$. The covariant derivatives are

$$D_{a\alpha} = \partial_{a\alpha} + i(\sigma^\mu\bar\theta_a)_\alpha\partial_\mu\,, \qquad \bar D_{\dot\alpha}^a = \bar\partial_{\dot\alpha}^a + i(\theta^a\sigma^\mu)_{\dot\alpha}\partial_\mu\,. \tag{6.7}$$

The supersymmetry transformation is given by the following real differential operator acting on any superfield,

$$i(\varepsilon^a Q_a + \bar Q^a\bar\varepsilon_a) = i\varepsilon^{a\alpha}\partial_{a\alpha} - i\bar\varepsilon_a^{\dot\alpha}\bar\partial_{\dot\alpha}^a + (\theta^a\sigma^\mu\bar\varepsilon_a + \varepsilon^a\sigma^\mu\bar\theta_a)\partial_\mu\,, \tag{6.8}$$

where $\varepsilon^{a\alpha}$, $\bar\varepsilon_a^{\dot\alpha} = (\varepsilon^{a\alpha})^\dagger$ are the supersymmetry transformation parameters.

The conventional chiral superfield is defined to satisfy $\bar D_{\dot\alpha}^a\Phi = 0$ so that it is a function of $x_+^\mu = x^\mu + i\theta^a\sigma^\mu\bar\theta_a$ and $\theta^{a\alpha}$ only, $\Phi(x_+,\theta)$. However, in order to incorporate the noncommutativity into the chiral superspace such that the star product preserves the 'chirality', it is necessary to shift the complex variables, x_+^μ to the real ones, x^μ, and take $\Phi(x,\theta)$ as the newly defined "chiral" superfield satisfying $\bar\partial_{\dot\alpha}^a\Phi = 0$. We define the shifting operator,

$$J = e^{-i\theta^a\sigma^\mu\bar\theta_a\partial_\mu}\,, \tag{6.9}$$

to satisfy $J^\dagger = J^{-1}$, $J\Phi(x_+,\theta) = \Phi(x,\theta)$, $J(\Phi\Phi') = (J\Phi)(J\Phi')$, and $J\bar D_{\dot\alpha}^a J^\dagger = \bar\partial_{\dot\alpha}^a$. The supersymmetry transformation of the chiral superfield is then,

$$\delta\Phi(x,\theta) = \mathcal{Q}\Phi(x,\theta)\,, \qquad \mathcal{Q} \equiv iJ(\varepsilon^a Q_a + \bar Q^a\bar\varepsilon_a)J^\dagger = i\varepsilon^{a\alpha}\partial_{a\alpha} - i\bar\varepsilon_a^{\dot\alpha}\bar\partial_{\dot\alpha}^a + 2\theta^a\sigma^\mu\bar\varepsilon_a\partial_\mu\,. \tag{6.10}$$

Similarly the anti-chiral superfield, $\bar\Phi(x,\bar\theta) = J^\dagger\bar\Phi(x_-,\bar\theta)$, transforms as

$$\delta\bar\Phi(x,\bar\theta) = \bar{\mathcal{Q}}\bar\Phi(x,\bar\theta)\,, \qquad \bar{\mathcal{Q}} \equiv iJ^\dagger(\varepsilon^a Q_a + \bar Q^a\bar\varepsilon_a)J = i\varepsilon^{a\alpha}\partial_{a\alpha} - i\bar\varepsilon_a^{\dot\alpha}\bar\partial_{\dot\alpha}^a + 2\varepsilon^a\sigma^\mu\bar\theta_a\partial_\mu\,. \tag{6.11}$$

As an example we consider the $\mathcal{N}=1$ chiral scalar theory [25]. Acting J on the standard lagrangian gives rise to only a total derivative term. Thus, we can rewrite the chiral superfield action in a different but equivalent form which is ready to incorporate the noncommutativity on the superspace.[5] From $J[\Phi(x_+,\theta)\bar\Phi(x_-,\bar\theta)] = \Phi(x,\theta)J^2\bar\Phi(x,\bar\theta)$, the action reads

$$\int dx^4 d\theta^2 d\bar\theta^2\,\Phi(x,\theta)J^2\bar\Phi(x,\bar\theta) + \int dx^4 d\theta^2\,W(\Phi(x,\theta)) + \int dx^4 d\bar\theta^2\,\overline{W}(\bar\Phi(x,\bar\theta))\,, \tag{6.12}$$

where explicitly,

$$J^2 = 1 - 2i\theta\sigma^\mu\bar\theta\partial_\mu - \theta^2\bar\theta^2\Box\,. \tag{6.13}$$

From $\mathcal{Q}J^2 = J^2\bar{\mathcal{Q}}$, the supersymmetry is manifest.

[5] This kind of rewriting the action was first considered in [15] which also includes the gauge multiplet. Our point here is to show that the full supersymmetry can be unbroken for the scalar theory on the minimally noncommutative superspace. Including the gauge multiplet appears to break the supersymmetry. It would be very interesting to find a mechanism not to break the supersymmetry in that case too.

Before replacing the ordinary products by the star products, it is crucial to note that from (3.7) we can rewrite $J^2\bar{\Phi}(x,\bar{\theta})$ in terms of the star product,

$$J^2\bar{\Phi} = \bar{\Phi} - \frac{1}{2} i\sigma^{\mu}_{\alpha\dot{\alpha}}[\theta^{\alpha},\{\bar{\theta}^{\dot{\alpha}},\partial_{\mu}\bar{\Phi}\}_{\star}]_{\star} - \frac{1}{16}[\theta^{\alpha},\{\theta_{\alpha},[\bar{\theta}_{\dot{\alpha}},\{\bar{\theta}^{\dot{\alpha}},\Box\bar{\Phi}\}_{\star}]_{\star}\}_{\star}]_{\star}\,. \tag{6.14}$$

With this understanding, we insert the star products into the action such that $\Phi\star(J^2\bar{\Phi})$, $W_{\star}(\Phi)$, $\overline{W}_{\star}(\bar{\Phi})$. Up to the total derivative term, the kinetic term remains same as in the commutative case, while the potential terms generically get nontrivial star product deformations but maintaining the 'chiral' property. The map to the supermatrix model is then straightforward. We merely mention that the size of the gamma matrices for the odd coordinates must be 4×4 for the nondegenerate case, $\det S \neq 0$ resulting in the generic supermatrix model (cf. [15]). The notion of 'chiral' supermatrix is valid as $[\lambda^2\hat{\bar{\partial}}_{\dot{\alpha}},\hat{\Phi}] = 0$.

From (6.10) and (6.11) it is clear that the supersymmetry is broken completely for the generic case, $c^{\alpha\beta} = (\bar{c}^{\dot{\alpha}\dot{\beta}})^{*} \neq 0$. However, for the minimally noncommutative superspace, where $c^{\alpha\beta} = 0$, as $\mathcal{Q}$ satisfies the Leibniz rule acting on the 'chiral' superfields, the full supersymmetry is unbroken.[6]

To consider the commutative limit, we take the noncommutative chiral superspace, (4.10), with the replacement, $(\theta^{\mu\nu}, c^{\alpha\beta}, \tilde{c}^{\alpha\dot{\beta}}, \bar{c}^{\dot{\alpha}\dot{\beta}}) \rightarrow (\lambda^2\theta^{\mu\nu}, \lambda^4 c^{\alpha\beta}, \lambda^4\tilde{c}^{\alpha\dot{\beta}}, \lambda^4\bar{c}^{\dot{\alpha}\dot{\beta}})$. In the commutative limit, $\lambda\rightarrow 0$, $\hat{x}^{\mu}\sim\lambda$, $\hat{\theta}^{\alpha}\sim\lambda^2$, $\partial_{\mu}\sim\lambda^{-1}$, $\partial_{\alpha}\sim\lambda^{-2}$, but $\det(\lambda^2\theta)/\det(\lambda^4\tilde{c}) = \det(\theta)/\det(\tilde{c})$ remains fixed. Thus, the kinetic part diverges while the superpotential term is finite. Calculating the partition function (5.3), it is natural to use the saddle point approximation with the constraint on the supermatrix, $\partial^{\alpha}\partial_{\alpha}\hat{\Phi} = \partial_{\alpha}\partial_{\mu}\hat{\Phi} = 0$, and hence $\hat{\Phi} = 1_{4\times 4}\hat{\phi}(\hat{x}/\lambda) + \lambda^{-2}\hat{\theta}^{\alpha}\psi_{\alpha}$ where ψ_{α} is a constant Grassmann variable. The whole kinetic term vanishes then in the supermatrix model. The supersymmetry is restored, from (3.6), (3.7), (4.6), (4.9), (6.10), as the adjoint action,

$$\delta\hat{\Phi} = i\left[\begin{pmatrix}\varepsilon^{\alpha} & 0\\ 0 & -\varepsilon^{\alpha}\end{pmatrix}\lambda^2\hat{\partial}_{\alpha}\,,\,\hat{\Phi}\right]\,, \tag{6.15}$$

which preserves the constraint, $\partial^{\alpha}\partial_{\alpha}\hat{\Phi} = \partial_{\alpha}\partial_{\mu}\hat{\Phi} = 0$.

7. Comments

It is straightforward to generalize the above analysis to the $\mathcal{N}$-extended supersymmetry in the minkowskian four dimensions. The commutative limit keeping the chiral superpotential finite is given by $\hat{x}^{\mu}\sim\lambda^{\mathcal{N}}$, $\hat{\theta}^{\alpha}\sim\lambda^2$ so that $Q_{\alpha}\sim\lambda^{-2}(1+\lambda^{4-\mathcal{N}})$, where the zeroth order inside the bracket comes from ∂_{α} while the $(4-\mathcal{N})$th order is from the term, $\theta\partial_{\mu}$, which breaks the Leibniz rule. The overall factor, λ^{-2}, can be absorbed into the supersymmetry parameter. Thus only for $\mathcal{N} = 1,2,3$ cases the supersymmetry can be restored for the dual supermatrix model in the limit. This is consistent with the rigidity of the $\mathcal{N} = 4$ theory or the absence of the superpotential in the theory. Furthermore, the commutative limit to keep the integral over the full superspace finite is $\hat{x}^{\mu}\sim\lambda^{\mathcal{N}}$, $\hat{\theta}^{\alpha}\sim\lambda$ so that $Q_{\alpha}\sim\lambda^{-1}(1+\lambda^{2-\mathcal{N}})$. Thus, the $\mathcal{N} = 4$ case would not restore the supersymmetry again. We take this as an evidence that there should be no $\mathcal{N} = 4$ superfield formalism.

[6] For the invariance of the bosonic theory under the scaling of the noncommutativity parameter, see [26].

One may conceive a theory on the noncommutative superspace which replaces all the covariant derivatives, D_α, in the conventional superfield theory by the ordinary derivative, ∂_α, basically dropping the $\vartheta^\alpha\partial_\mu$ term. The full supersymmetry is simply given by the derivative along the odd coordinates and is obviously unbroken, since $[\hat{\partial}_\alpha, [\hat{\partial}_A, \hat{F}\}_-\}_- = (-1)^{\#_A}[\hat{\partial}_A, [\hat{\partial}_\alpha, \hat{F}\}_-\}_-$. Disregarding the spacetime derivatives in the commutative supersymmetric theory would lead to a trivial kinetic term for the component fields. However, on the noncommutative superspace it seems not the case. After all, the star product already contains infinitely many derivatives.

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A. The proof of the isomorphism

Here we prove the isomorphism (3.5) between the operator and the star product formalisms by the mathematical induction on n with $f_n = z^{A_1}\cdots z^{A_n}$. Firstly we note for arbitrary $m \geq 0$, $g_m = z^{B_1}\cdots z^{B_m}$,

$$
\begin{aligned}
&\hat{z}^A\mathcal{O}(g_m) - \mathcal{O}(z^A g_m) = \\
&= \sum_{P=1}^{m!}\frac{(-1)^{N_P}}{(m+1)!}\sum_{j=1}^{m+1}\left[\hat{z}^A\hat{z}^{P_1}\hat{z}^{P_2}\cdots\hat{z}^{P_m} - (-1)^{\#_A(\#_{P_1}+\cdots+\#_{P_{j-1}})}\hat{z}^{P_1}\cdots\hat{z}^{P_{j-1}}\hat{z}^A\hat{z}^{P_j}\cdots\hat{z}^{P_m}\right] \\
&= \sum_{P=1}^{m!}\frac{(-1)^{N_P}}{(m+1)!}\sum_{j=1}^{m}[\hat{z}^A, \hat{z}^{P_1}\cdots\hat{z}^{P_j}\}\hat{z}^{P_{j+1}}\cdots\hat{z}^{P_m} \\
&= \sum_{P=1}^{m!}\frac{(-1)^{N_P}}{(m+1)!}\sum_{j=1}^{m}\left(\sum_{l=1}^{j}(-1)^{\#_A(\#_{P_1}+\cdots+\#_{P_{l-1}})}\hat{z}^{P_1}\cdots\hat{z}^{P_{l-1}}[\hat{z}^A, \hat{z}^{P_l}\}\hat{z}^{P_{l+1}}\cdots\hat{z}^{P_j}\right)\hat{z}^{P_{j+1}}\cdots\hat{z}^{P_m} \\
&= \sum_{P=1}^{m!}\frac{(-1)^{N_P}}{(m+1)!}\sum_{j=1}^{m}(m+1-j)(-1)^{\#_A(\#_{P_1}+\cdots+\#_{P_{j-1}})}\hat{z}^{P_1}\cdots\hat{z}^{P_{j-1}}[\hat{z}^A, \hat{z}^{P_j}\}\hat{z}^{P_{j+1}}\cdots\hat{z}^{P_m} \\
&= \sum_{P=1}^{m!}\frac{(-1)^{N_P}}{(m+1)!}\sum_{j=1}^{m}(m+1-j)(-1)^{\#_{P_j}(\#_{P_1}+\cdots+\#_{P_{j-1}})}[\hat{z}^A, \hat{z}^{P_j}\}\hat{z}^{P_1}\cdots\cancel{\hat{z}}^{P_j}\cdots\hat{z}^{P_m} \\
&= \tfrac{1}{2}\sum_{k=1}^{m}(-1)^{\#_{B_k}(\#_{B_1}+\cdots+\#_{B_{k-1}})}[\hat{z}^A, \hat{z}^{B_k}\}\mathcal{O}(z^{B_1}\cdots\cancel{z}^{B_k}\cdots z^{B_m}) \\
&= \mathcal{O}\left(\frac{1}{2}\Omega^{AC}\partial_C g_m\right).
\end{aligned}
\tag{A.1}
$$

This shows $\mathcal{O}(\hat{z}^A)\mathcal{O}(g) = \mathcal{O}(z^A \star g)$ for arbitrary g. Now we assume $\mathcal{O}(f_k)\mathcal{O}(g) = \mathcal{O}(f_k \star g)$ for $k \leq n-1$. From

$$\mathcal{O}(f_n) = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_j \, \hat{z}^{A_j} \mathcal{O}(z^{A_1} \cdots \not{z}^{A_j} \cdots z^{A_n}) \,, \qquad \varepsilon_j = (-1)^{\#_{A_j}(\#_{A_1}+\cdots+\#_{A_{j-1}})} \,, \tag{A.2}$$

it follows

$$\mathcal{O}(f_n)\mathcal{O}(g) = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_j \, \mathcal{O}\left(z^{A_j} \star [(z^{A_1} \cdots \not{z}^{A_j} \cdots z^{A_n}) \star g]\right) . \tag{A.3}$$

On the other hand, a straightforward manipulation gives for any $1 \leq j \leq n$,

$$\begin{aligned}
&\left(z^{A_1} \cdots z^{A_n}\right) \left(\frac{1}{2}\overleftrightarrow{\Omega}\right)^m g = \\
&= \varepsilon_j \left(z^{A_j} z^{A_1} \cdots \not{z}^{A_j} \cdots z^{A_n}\right) \left(\frac{1}{2}\overleftrightarrow{\Omega}\right)^m g \\
&= \varepsilon_j \left[\begin{array}{l}
z^{A_j} \left\{\left(z^{A_1} \cdots \not{z}^{A_j} \cdots z^{A_n}\right) \left(\frac{1}{2}\overleftrightarrow{\Omega}\right)^m g\right\} \\[1em]
+m\frac{1}{2}\Omega^{A_j B}\partial_B \left\{\left(z^{A_1} \cdots \not{z}^{A_j} \cdots z^{A_n}\right) \left(\frac{1}{2}\overleftrightarrow{\Omega}\right)^{m-1} g\right\} \\[1em]
-m\frac{1}{2}\sum_{l<j} \varepsilon_l \Omega^{A_j A_l} \left\{\left(z^{A_1} \cdots \not{z}^{A_l} \cdots \not{z}^{A_j} \cdots z^{A_n}\right) \left(\frac{1}{2}\overleftrightarrow{\Omega}\right)^{m-1} g\right\} \\[1em]
-m\frac{1}{2}\sum_{l>j} \varepsilon_l (-1)^{\#_{A_l}\#_{A_j}} \Omega^{A_j A_l} \left\{\left(z^{A_1} \cdots \not{z}^{A_j} \cdots \not{z}^{A_l} \cdots z^{A_n}\right) \left(\frac{1}{2}\overleftrightarrow{\Omega}\right)^{m-1} g\right\}
\end{array}\right] .
\end{aligned} \tag{A.4}$$

From (2.4), summing over j cancels the last two terms so that

$$(z^{A_1} \cdots z^{A_n}) \star g = \frac{1}{n}\sum_{j=1}^{n} \varepsilon_j \left(z^{A_j} + \frac{1}{2}\Omega^{A_j B}\partial_B\right) \left\{(z^{A_1} \cdots \not{z}^{A_j} \cdots z^{A_n}) \star g\right\} . \tag{A.5}$$

Thus, combining with (A.3) we get $\mathcal{O}(f_n)\mathcal{O}(g) = \mathcal{O}(f_n \star g)$, which completes our proof.

References

[1] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[2] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring*, *Nucl. Phys.* **B 498** (1997) 467 [hep-th/9612115].

[3] W. Taylor, *M(atrix) theory: matrix quantum mechanics as a fundamental theory*, *Rev. Mod. Phys.* **73** (2001) 419 [hep-th/0101126].

[4] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048].

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[5] S. Hyun and J.-H. Park, *5d action for longitudinal five branes on a PP-wave*, *J. High Energy Phys.* **11** (2002) 001 [hep-th/0209219];
S. Hyun, J.-H. Park and S.-H. Yi, *3d $N = 2$ massive super Yang-Mills and membranes/D2-branes in a curved background*, *J. High Energy Phys.* **03** (2003) 004 [hep-th/0301090].

[6] A.P. Polychronakos, *Generalized statistics in one dimension*, hep-th/9902157; *Integrable systems from gauged matrix models*, *Phys. Lett.* **B 266** (1991) 29.

[7] J.-H. Park, *Description of identical particles via matrix model: generalization of sutherland system*, *Phys. Lett.* **A 307** (2003) 183 [hep-th/0203017].

[8] T. Eguchi and H. Kawai, *Reduction of dynamical degrees of freedom in the large-N gauge theory*, *Phys. Rev. Lett.* **48** (1982) 1063;
G. Parisi, *A simple expression for planar field theories*, *Phys. Lett.* **B 112** (1982) 463;
D.J. Gross and Y. Kitazawa, *A quenched momentum prescription for large-N theories*, *Nucl. Phys.* **B 206** (1982) 440;
G. Bhanot, U.M. Heller and H. Neuberger, *The quenched Eguchi-Kawai model*, *Phys. Lett.* **B 113** (1982) 47;
S.R. Das and S.R. Wadia, *Translation invariance and a reduced model for summing planar diagrams in QCD*, *Phys. Lett.* **B 117** (1982) 228.

[9] A. Gonzalez-Arroyo and M. Okawa, *The twisted Eguchi-Kawai model: a reduced model for large-N lattice gauge theory*, *Phys. Rev.* **D 27** (1983) 2397.

[10] R. Dijkgraaf and C. Vafa, *Matrix models, topological strings and supersymmetric gauge theories*, *Nucl. Phys.* **B 644** (2002) 3 [hep-th/0206255]; *On geometry and matrix models*, *Nucl. Phys.* **B 644** (2002) 21 [hep-th/0207106]; *A perturbative window into non-perturbative physics*, hep-th/0208048.

[11] D. Klemm, S. Penati and L. Tamassia, *Non(anti)commutative superspace*, *Class. and Quant. Grav.* **20** (2003) 2905 [hep-th/0104190].

[12] H. Ooguri and C. Vafa, *The c-deformation of gluino and non-planar diagrams*, hep-th/0302109.

[13] N. Seiberg, *Noncommutative superspace, $N = 1/2$ supersymmetry, field theory and string theory*, *J. High Energy Phys.* **06** (2003) 010 [hep-th/0305248].

[14] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, hep-th/0302078.

[15] H. Kawai, T. Kuroki and T. Morita, *Dijkgraaf-Vafa theory as large-N reduction*, *Nucl. Phys.* **B 664** (2003) 185 [hep-th/0303210].

[16] I. Chepelev and C. Ciocarlie, *A path integral approach to noncommutative superspace*, *J. High Energy Phys.* **06** (2003) 031 [hep-th/0304118].

[17] R. Britto, B. Feng and S.-J. Rey, *Deformed superspace, $N = 1/2$ supersymmetry and (non)renormalization theorems*, *J. High Energy Phys.* **07** (2003) 067 [hep-th/0306215].

[18] S. Terashima and J.-T. Yee, *Comments on noncommutative superspace*, hep-th/0306237.

[19] M. Hatsuda, S. Iso and H. Umetsu, *Noncommutative superspace, supermatrix and lowest Landau level*, hep-th/0306251.

[20] S. Ferrara, M.A. Lledo and O. Macia, *Supersymmetry in noncommutative superspaces*, `hep-th/0307039`.

[21] I.L. Buchbinder and S.M. Kuzenko, *Ideas and methods of supersymmetry and supergravity: a walk through superspace*, IOP Publishing, Bristol 1995.

[22] D. Bak, K.-M. Lee and J.-H. Park, *Comments on noncommutative gauge theories*, *Phys. Lett.* **B 501** (2001) 305 [`hep-th/0011244`].

[23] E. Langmann, R.J. Szabo and K. Zarembo, *Exact solution of noncommutative field theory in background magnetic fields*, *Phys. Lett.* **B 569** (2003) 95 [`hep-th/0303082`].

[24] S.J. Gates, M.T. Grisaru, M. Roček and W. Siegel, *Superspace, or one thousand and one lessons in supersymmetry*, *Front. Phys.* **58** (1983) 1 [`hep-th/0108200`].

[25] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton 1992.

[26] T. Ishikawa, S.-I. Kuroki and A. Sako, *Noncommutative cohomological field theory and GMS soliton*, *J. Math. Phys.* **43** (2002) 872 [`hep-th/0107033`].

JHEP09(2003)046